# Elements of Computational Learning Theory

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# Outline

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# What is Machine Learning?

- **Machine learning** is the subfield of computer science that gives "computers the ability to learn without being explicitly programmed".

  - term coined by Arthur Samuel in 1959 while at IBM

- The study of algorithms that can learn from data.

# Another View of Machine Learning

- Learning from historical data to make decisions about unseen data.

- Traditional Programming

  Data $\longrightarrow$
  Program $\longrightarrow$ [Computer] $\longrightarrow$ Output

- Machine Learning

  Data $\longrightarrow$
  Output $\longrightarrow$ [Computer] $\longrightarrow$ Program

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    - humans can not describe how they do a task
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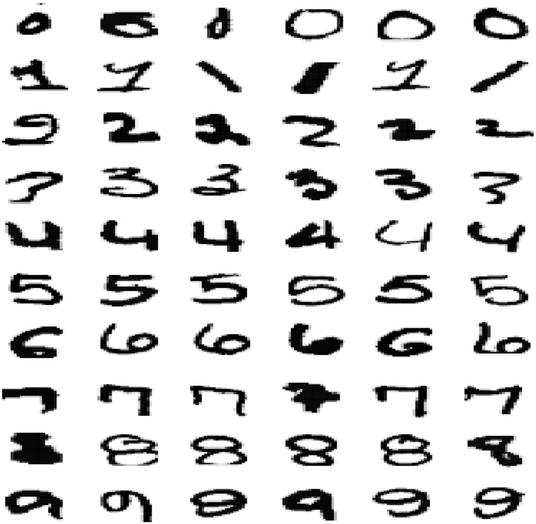
  - the desired function changes frequently
    - recommend stock transactions
  - each user needs a customized function $f$
    - email spam / ham
    - email importance (perhaps delete without presenting?)
    - recommendations on Amazon

# Can you write a program that recognizes these digits?



# What Machine Learning Does

- Class A

- Class B

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- Want to be able to generalize the classification to unseen data.

http://ciml.info/ (Credit: Hal Daumé III)

# Classify These

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# Let's See ...

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- Bird vs non-bird

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- Flies vs not-flies

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- We need **bias** in order to be able to generalize to unseen data.

# No Free-Lunch Theorems

### Theorem 1

*Let $\mathcal{F}$ be the set of all possible Boolean functions on n variables. Let $Acc_G(L)$ be the (generalization) accuracy of L on non-training examples. Then, for any consistent learner L, it holds*

$$\frac{1}{|\mathcal{F}|} \cdot \sum_{\mathcal{F}} Acc_G(L) = 1/2 \,.$$

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### Proof Sketch.

Let $S$ be the set of training examples.
Let $f \in \mathcal{F}$ such that $Acc_G(f) = \frac{1}{2} + \delta$.
Then, $\exists f' \in \mathcal{F}$ such that $Acc_G(f') = \frac{1}{2} - \delta$.
To see why, note that we can have an $f' \in \mathcal{F}$ that satisfies:

$$\begin{cases} (\forall x \in S)(f'(x) = f(x)) \\ (\forall x \notin S)(f'(x) = \neg f(x)) \end{cases} \quad \square$$

# No Free-Lunch Theorems

**Theorem 2**

*Let $\mathcal{F}$ be the set of all possible Boolean functions on $n$ variables. Let $Acc_G(L)$ be the (generalization) accuracy of $L$ on non-training examples. Then, for any consistent learner $L$, it holds*

$$\frac{1}{|\mathcal{F}|} \cdot \sum_{\mathcal{F}} Acc_G(L) = {}^1\!/_2\,.$$

**Corollary 3**

*For any two learners $L_1, L_2$, if there exists a learning problem $P$ such that $Acc_G(L_1) > Acc_G(L_2)$, then there exists another learning problem $P'$ such that $Acc_G(L_1) < Acc_G(L_2)$.*

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- How frequently will our solution make mistakes during prediction?
    - How confident are we about such a claim?

# The Main Goal of Computational Learning Theory

Find a good approximation of a function with high probability

# Computational Learning Theory

**Goal** (Good Approximation with High Probability)
There is a function c over a space $X$. One wants to come up (in a reasonable amount of time) with a function h such that h is a *good approximation* of c with *high probability*.

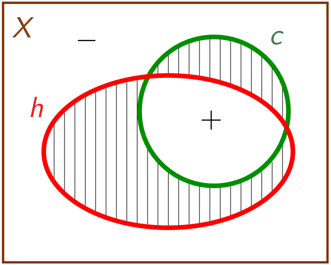
## Description 1 (Parameters and Terminology)

- $X$: *Instance Space* (*say*, $\{0, 1\}^n$) $\mathcal{Y}$: *Labels* (*say*, $\{+, -\}$)
- $c \in \mathcal{C}$: *Target concept belonging to a concept class*
- $h \in \mathcal{H}$: *Hypothesis belonging to a hypothesis class*
- *Good Approximation: Small Risk (Error)* $\varepsilon$
- *High Probability: Confidence* $1 - \delta$
- *Reasonable Amount of Time: Polynomial in* $n, {}^{1}/_{\varepsilon}, {}^{1}/_{\delta}$, *size*($c$)

- Realizability assumption: $(\forall c \in \mathcal{C})(\exists h \in \mathcal{H})(\forall x \in X)\,[h(x) = c(x)]$
($\mathcal{H}$ is at least as expressive as $\mathcal{C}$; we will see an example later)

# Probably Approximately Correct (PAC) Learning

- There is an *arbitrary, unknown* distribution $\mathcal{D}$ over $X$.
- Learn from *poly* $\left(\frac{1}{\varepsilon}, \frac{1}{\delta}\right)$ many examples $(x, \mathrm{c}(x))$, where $x \sim \mathcal{D}$.
- $\mathrm{Risk}_{\mathcal{D}}(\mathrm{h}, \mathrm{c}) = \mathrm{Pr}_{x\sim\mathcal{D}}(\mathrm{h}(x) \neq \mathrm{c}(x))$.




**Goal 1 ([Valiant, 1984])**

$$\Pr\left(Risk_{\mathcal{D}}(h, c) \leq \varepsilon\right) \geq 1 - \delta\ .$$

# Typical Functions Used for Learning

Monotone Conjunctions/Monomials (Boolean AND of some variables chosen from $\{x_1, x_2, \ldots, x_n\}$)

e.g., $c = x_2 \wedge x_5 \wedge x_8$ (sometimes simply write $c = x_2x_5x_8$)

- $|\mathcal{H}| = 2^n$.
- Sometimes we may need to include the `FALSE` function (e.g., for VC-dimension arguments) even if such a function can not be represented by combining variables. In these cases $|\mathcal{H}| = 2^n + 1$.

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Halfspaces $\quad$ e.g., $c = sgn(w_0 + w_1 \cdot x_1 + w_2 \cdot x_2 + \ldots + w_n \cdot x_n)$

$$sgn(z) = \begin{cases} +1 & \text{, if } z > 0 \\ -1 & \text{, if } z \leq 0 \end{cases}$$

# Why These Functions Used as Toy Examples?

- Exhibit **bias**.
- (Monotone) conjunctions is one of the most basic ways of selecting/combining features/constraints in a prediction mechanism.
- Building blocks for richer classes of functions that are less understood; e.g., general DNF formulae.
  (e.g., learning monotone DNF formulae over the uniform distribution is an open problem.)
- Directly or indirectly, applications to logic, circuit complexity, etc.
- Typical benchmarks as they usually provide interesting, but non-trivial insights of the definitions, the bounds that we should expect to get, etc.
- Can also be useful in contexts of other disciplines (see next slide).

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- Can also be useful in contexts of other disciplines (see next slide).
- We will start with PAC learning (general) conjunctions.

# Finding All Common Properties of a Set of Objects

Let $X = \{0,1\}^6$ and $c = x_1 \wedge \overline{x}_3 \wedge x_4$.

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- Delete the variables that are falsified by the positive examples.

*A Study of Thinking* [Bruner, Goodnow, Austin, 1956]

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- Is such an algorithm good for PAC learning?
  - YES, provided $m$ is large enough.
  - Creates a consistent hypothesis:
    - Predicts correct label for each training example.

# How Many Examples are Enough?

**Theorem 4 (PAC Learning of Finite Concept Classes)**

*Let $\mathcal{H}$ contain a* ***finite amount*** $|\mathcal{H}|$ *of functions. For every distribution $\mathcal{D}$, drawing $m \geq \frac{1}{\varepsilon} \cdot \left(\ln|\mathcal{H}| + \ln\frac{1}{\delta}\right)$ examples are* ***enough*** *to guarantee that* ***any consistent*** *hypothesis $h$ satisfies* $\Pr\left(Risk_{\mathcal{D}}(h, c) \leq \varepsilon\right) \geq 1 - \delta$ .

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- Return some hypothesis that never made a mistake. □

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Let $h_1, h_2, \ldots, h_k$ be all the $k$ hypotheses from $\mathcal{H}$ that are bad. For each such bad hypothesis $h_i$ with $i \in \{1, \ldots, k\}$, consider the bad event

$$B_i \equiv h_i \text{ is consistent with all } m \text{ training examples}\,.$$

$\Pr\left(B_1 \vee \ldots \vee B_k\right) \leq \sum_{i=1}^{k} \Pr\left(B_i\right) < k \cdot (1-\varepsilon)^m \leq |\mathcal{H}|\,(1-\varepsilon)^m \leq |\mathcal{H}| \cdot e^{-\varepsilon \cdot m}$. □

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**Theorem 6 (PAC Learning of Finite Concept Classes)**

*Let $\mathcal{H}$ be a hypothesis space s.t. $|\mathcal{H}| < \infty$. For every distribution $\mathcal{D}$, drawing $m \geq \frac{1}{\varepsilon} \cdot \left( \ln|\mathcal{H}| + \ln\frac{1}{\delta} \right)$ examples are* ***enough*** *to guarantee that* ***any consistent*** *hypothesis $h$ satisfies $\Pr\left(Risk_{\mathcal{D}}(h, c) \leq \varepsilon\right) \geq 1 - \delta$ .*

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  - $|\mathcal{H}| = 3^n + 1 \quad \leq \quad 3^n + 3^n + 3^n = 3^{n+1}$
  - Theorem imples that when $n = 20$ ($2^{20}$ truth assignments in $\{0, 1\}^{20}$) only $m = 2607$ examples are enough! (less than 0.2% of $|X|$)

# Can we Learn a Disjunction of $k \geq 2$ Conjunctions?

- Say $k = 3$. Then a function looks like
  $(x_1 \wedge x_5) \vee (\overline{x_2} \wedge x_4 \wedge x_7) \vee (x_3 \wedge \overline{x}_4 \wedge \overline{x_5} \wedge x_7 \wedge \overline{x_8})$.
- Then, $|\mathcal{C}| \leq (3^n + 1) \cdot (3^n + 1) \cdot (3^n + 1) \leq 3^{n+1} \cdot 3^{n+1} \cdot 3^{n+1} = 3^{3n+3}$.
- The previous theorem implies $m = \left\lceil \frac{1}{\varepsilon} \cdot \ln\left(\frac{3^{3n+3}}{\delta}\right)\right\rceil = \left\lceil \frac{3n+3}{\varepsilon} \cdot \ln\left(\frac{3}{\delta}\right)\right\rceil$ training examples are more than enough for PAC learning the class.

So the question becomes:

Is there an algorithm for efficiently PAC learning such functions?
The answer is quite surprising!

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- Assuming $NP \neq RP$, we can not do that efficiently if we use $\mathcal{H} = \mathcal{C}$. (proper learning)
- However, we can PAC learn $\mathcal{C}$ efficiently if we use a larger class of functions as our hypothesis class $\mathcal{H}$.

# The Complexity Class *RP*

**Randomized Polynomial (*RP*) time.** Complexity class of problems for which a non-deterministic Turing machine:

- runs in poly-time w.r.t. the input size,
- if the correct answer is *NO* it returns *NO*,
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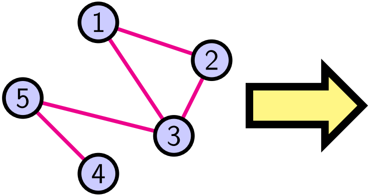
- Let us return to our problem now.

# An Intractability Result

**Theorem 7**

*If RP $\neq$ NP, the representation class of $k$-term DNF formulae is not efficiently PAC learnable for any $k \geq 2$.*

**Proof Idea:** Reduce Graph 3-Coloring problem to the problem of finding a consistent 3-term DNF formula with a sample $S_G = S_G^+ \cup S_G^-$.




$\langle(0,1,1,1,1),+\rangle$ $\langle(0,0,1,1,1),-\rangle$
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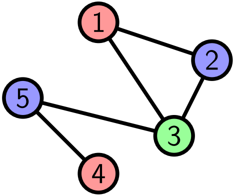
$\langle(1,1,1,0,0),-\rangle$

- Positive examples encode the vertices of the given graph.
- Negative examples encode the edges of the given graph.
- Show: $G$ is 3-colorable *iff* $S_G$ is consistent with some 3-term DNF.

## $G$ is 3-colorable $\Rightarrow$ $S_G$ consistent with some 3-term DNF



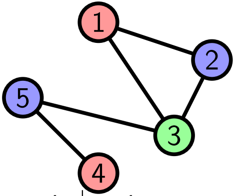

| node | color |
|---|---|
| 1 | *red* |
| 2 | *blue* |
| 3 | *green* |
| 4 | *red* |
| 5 | *blue* |

$$\begin{array}{ll} \langle(0,1,1,1,1),+\rangle & \langle(0,0,1,1,1),-\rangle \\ \langle(1,0,1,1,1),+\rangle & \langle(0,1,0,1,1),-\rangle \\ \langle(1,1,0,1,1),+\rangle & \langle(1,0,0,1,1),-\rangle \\ \langle(1,1,1,0,1),+\rangle & \langle(1,0,1,1,0),-\rangle \\ \langle(1,1,1,1,0),+\rangle & \langle(1,1,0,1,0),-\rangle \\ & \langle(1,1,1,0,0),-\rangle \end{array}$$

$$\Longrightarrow \left\{ \begin{array}{lcl} T_r & = & x_2 \wedge x_3 \wedge x_5 \\ T_b & = & x_1 \wedge x_3 \wedge x_4 \\ T_g & = & x_1 \wedge x_2 \wedge x_4 \wedge x_5 \end{array} \right. \Longrightarrow \varphi = T_r \vee T_b \vee T_g$$

- Consider a positive example $v(i) \in S_G^+$. Let $color(node\ i) = red$ (similar argument for other colors). Then, $T_r$ is a conjunction of non-red nodes, so $v(i)$ satisfies $T_r$ (and therefore $\varphi$).

# $G$ is 3-colorable $\Rightarrow$ $S_G$ consistent with some 3-term DNF

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- Let $e(i,j) \in S_G^-$. A valid 3-coloring with nodes $i$ and $j$ connected by an edge implies that they have a different color. But $e(i,j)$ will falsify at least one of the variables in the term (say $T_r$) since at least one of the two nodes must have color other than red and is therefore included in the term $T_r$.

# $S_G$ consistent with some 3-term DNF $\Rightarrow$ $G$ is 3-colorable

Let $\varphi = \boldsymbol{T_r} \vee \boldsymbol{T_b} \vee \boldsymbol{T_g}$ be consistent with $S_G$.

We claim that the following coloring is valid:

- color node $i$ red if $v(i) \in S_G^+$ satisfies $T_r$.
- color node $i$ blue if $v(i) \in S_G^+$ satisfies $T_b$.
- color node $i$ green if $v(i) \in S_G^+$ satisfies $T_g$.
- (break ties arbitrarily if $v(i) \in S_G^+$ satisfies more than one term)

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Since $\varphi$ is consistent with $S_G$, every $v(i) \in S_G^+$ satisfies some term $\Rightarrow$ every node is assigned a color.

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Since $\varphi$ is consistent with $S_G$, every $v(i) \in S_G^+$ satisfies some term $\Rightarrow$ every node is assigned a color.

- Suppose nodes $i$ and $j$ are assigned the same color (say red). Then both $v(i)$ and $v(j)$ satisfy term $T_r$. $\Rightarrow$ $x_i \notin T_r$ and moreover $\overline{x_i} \notin T_r$ because these two vectors satisfy $T_r$ and their $i$-th bit is 0 in one case and 1 in the other case.
- But $e(i,j)$ and $v(j)$ differ only in their $i$-th bit and if $v(j)$ satisfies $T_r$, so does $e(i,j)$. But then this means $e(i,j) \notin S_G^-$ since $\varphi$ is consistent with $S_G$. Therefore, $(i,j)$ is not an edge in $G$ as required.

# Why the Reduction is About *RP*?

- PAC learning should work for every small $\varepsilon$ and every small $\delta$.
- Work against this definition.
- If we have a sample $S$ of $m$ training examples (say, all distinct), a PAC learning algorithm should also be able to learn these $m$ examples to error $\varepsilon = \frac{1}{m+1}$ even when the distribution on these points is uniform; i.e., for every $(x, y) \in S$ it holds $\Pr_{x\sim\mathcal{D}}(x) = \frac{1}{m}$.

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- But then this means that the algorithm should create a **consistent** hypothesis with the training examples.

  (Otherwise the risk would be very large.)
- Per the PAC criterion, a consistent hypothesis will be created with high probability.
- This explains why we care about *RP*.

# Learning 3-Term DNF Formulae using 3-CNF Formulae

- We use the fact:

$$(u \wedge v) \vee (w \wedge z) = (u \vee w) \wedge (u \vee z) \wedge (v \vee w) \wedge (v \vee z)$$

- So, a 3-term DNF formula can be represented as a 3-CNF formula; i.e., a CNF formula where each clause has at most 3 literals.

$$T_1 \vee T_2 \vee T_3 = \bigwedge_{u \in T_1, v \in T_2, w \in T_3} (u \vee v \vee w)$$

- In general, this construction can take a $k$-term DNF formula and represent it with a $k$-CNF formula.
- Reduce the problem of learning a $k$-CNF formula to learning conjunctions:
  - For every triple $(u, v, w)$ over the original variables $\{x_1, \ldots, x_n\}$, create a variable $y_{u,v,w}$ corresponding to this triple.
  - Hence number of variables $y_{u,v,w}$ is at most $(2n)^3$, which is $O(n^3)$.
    (For k-term DNF the corresponding $y$'s will be $O(n^k)$ in total.)

## Learning 3-Term DNF Formulae using 3-CNF Formulae

- 3-CNF over $\{x_1, \ldots, x_n\}$ is equivalent to a 3-CNF over the new variables $\{y_{u,v,w}\}$.

So:

- A truth assignment $\sigma \in \{0,1\}^n$ corresponding to the variables $\{x_1, \ldots, x_n\}$ can be converted in time $O(n^3)$ to a truth assignment corresponding to the variables $\{y_{u,v,w}\}$.
- So, we can run our algorithm for learning conjunctions in polynomial time over the variables $\{y_{u,v,w}\}$.
    - Find-S may run in time $O(mn)$; for $m$ examples of bitsize $n$ each.
    - In the new setting: $n' \mapsto (2n)^3$ and $m' \approx O(n') = O(n^3)$.
- Once we are done learning, we can convert the solution that uses the variables $\{y_{u,v,w}\}$ back to $\{x_1, \ldots, x_n\}$ by simply expanding each variable $\{y_{u,v,w}\}$ to the clause $(u \vee v \vee w)$.

# Learning 3-Term DNF Formulae using 3-CNF Formulae

Finally, we need to argue that the solution that we compute indeed has low risk.

- Let $c$ be the target 3-CNF and $\mathcal{D}$ the target distribution over $\{0,1\}^n$.
- Let $c'$ be the target 3-CNF using the variables $\{y_{u,v,w}\}$ and $\mathcal{D}'$ the (induced) distribution over the assignments to the $\{y_{u,v,w}\}$ variables.
- We need to argue that if $h'$ has risk less than $\varepsilon$, so does $h$.
  - For $\sigma_1, \sigma_2 \in \{0,1\}^n$ with $\sigma_1 \neq \sigma_2$, it follows that we have $\sigma_1' \neq \sigma_2'$.
  - So, $h'(\sigma') \neq c'(\sigma') \Rightarrow$ there is a *unique preimage* $\sigma \in \{0,1\}^n$ such that $h(\sigma) \neq c(\sigma)$ and the weight of $\sigma$ under $\mathcal{D}$ is the same as that of $\sigma'$ under $\mathcal{D}'$.

(We have used the fact that *our algorithm learns under any distribution*.)

- For example, let $\mathcal{D}$ be the uniform distribution over $\{0,1\}^n$; i.e., each variable in the truth assignment is satisfied with probability $1/2$.
- Under $\mathcal{D}'$, a variable $y_{u,v,w}$ corresponding to the clause $(u \vee v \vee w)$ is satisfied with probability $7/8$. Similarly, $y_{u,u,u}$ is satisfied with probability $1/2$, or $y_{u,u,\overline{u}}$ is satisfied with probability $1$.

# How many examples are enough?

What if $|\mathcal{H}| = \infty$?

# Different Classifications and the Growth Function

- $\mathrm{x} = (x_1, x_2, \ldots, x_m)$ is a set of $m$ examples.

Number of Classifications $\Pi_{\mathcal{H}}(\mathrm{x})$ of x by $\mathcal{H}$: Distinct vectors $(\mathrm{h}(x_1), \mathrm{h}(x_2), \ldots, \mathrm{h}(x_m))$ as h runs through $\mathcal{H}$.

- $\Pi_{\mathcal{H}}(\mathrm{x}) \leq 2^m$.

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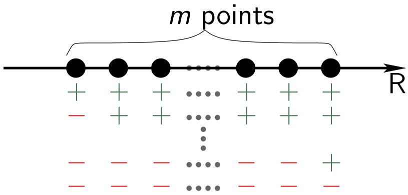
**Growth Function:** $\Pi_{\mathcal{H}}(m) = \max\{\Pi_{\mathcal{H}}(\mathrm{x}) \ : \ \mathrm{x} \in X^m\}$ .

### Example 8

Rays on a line:

$$\mathrm{h}_\vartheta(x) = \begin{cases} + & , \quad \text{if } x \geq \vartheta \\ - & , \quad \text{otherwise} \end{cases}$$

$\Pi_{\mathcal{H}}(m) = m + 1$ .



# The Vapnik-Chervonenkis Dimension

**Definition 9**

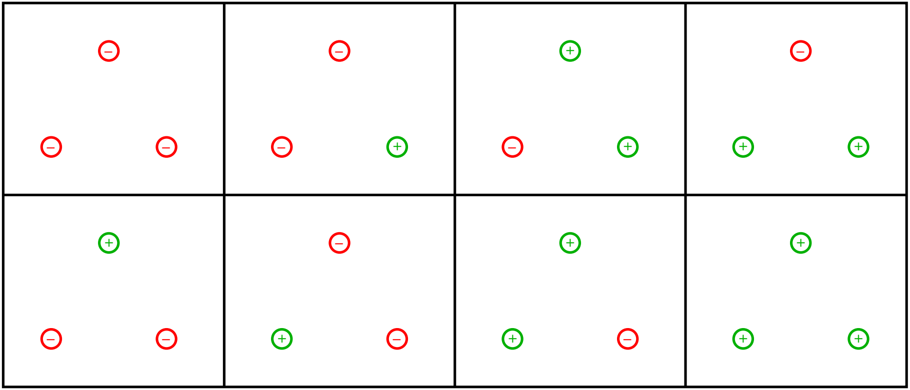
A sample x of size $m$ is *shattered* by $\mathcal{H}$, or $\mathcal{H}$ *shatters* x, if $\mathcal{H}$ can give all $2^m$ possible classifications of x.

**Definition 10 (*VC* dimension)**

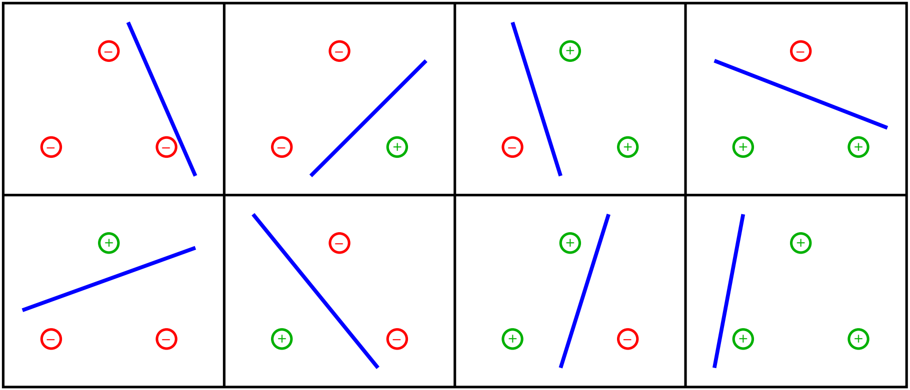
$$VC\text{-}dim(\mathcal{C}) = \max\{m \ : \ \Pi_{\mathcal{C}}(m) = 2^m\}$$

- Our ray example has $VC\text{-}dim(\text{Rays}) = 1$.
  - One point is shattered.
  - Two points are not shattered $(+, -)$
- Lower Bound $\Longrightarrow$ Explicit construction that achieves $2^m$.
- Upper Bound $\Longrightarrow$ For *any* sample x of length $m$ we can not achieve $2^m$.

# Configurations of 3 Points in 2D



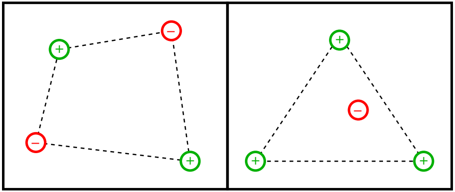
# Halfspaces Shatter 3 Points in 2D



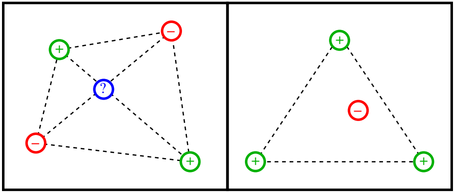

**Question 1**

*Can we shatter 4 points ?*

# Can Halfspaces Shatter 4 Points in 2D?



# Halfspaces can *not* Shatter 4 Points in 2D




### Theorem 11 (Radon)

*Any set of $d + 2$ points in $\mathbb{R}^d$ can be partitioned into two (disjoint) sets whose convex hulls intersect.*

### Corollary 12

- $VC\text{-}dim\,(HALFSPACES) = 3$ *in* $2$ *dimensions.*
- $VC\text{-}dim\,(HALFSPACES) = d + 1$ *in* $d \geq 1$ *dimensions.*

# Sauer's Lemma

**Lemma 13 (Sauer's Lemma)**

*Let $d \geq 0$ and $m \geq 1$ be given integers and let $\mathcal{H}$ be a hypothesis space with VC-dim$(\mathcal{H}) = d$. Then*

$$\Pi_{\mathcal{H}}(m) \leq 1 + \binom{m}{1} + \binom{m}{2} + \cdots + \binom{m}{d} = \Phi(d, m).$$

**Proposition 1**

*For all $m \geq d \geq 1$, $\Phi(d, m) < \left(\frac{em}{d}\right)^d$ .*

# VC Dimension: How Many Examples are Enough for Learning (Distribution Independently)?

**Theorem 14**

*Let $\mathcal{C}$ have finite VC-dim$(\mathcal{C}) = d \geq 1$ and moreover let $0 < \delta, \varepsilon < 1$. Then,*

$$m \geq \left\lceil \frac{4}{\varepsilon} \cdot \left( d \cdot \lg\left(\frac{12}{\varepsilon}\right) + \lg\left(\frac{2}{\delta}\right)\right)\right\rceil$$

*samples guarantee that any consistent hypothesis has small error with high probability (in the PAC-learning sense).*

- We still need an efficient algorithm to efficiently PAC-learn the class.

# VC Dimension: How Many Examples are Necessary for Learning (Distribution Independently)?

**Theorem 15**

*Any algorithm for PAC-learning a concept class of VC dimension $d$ with parameters $\varepsilon < 1/16$ and $\delta \leq 1/15$, must use*

$$m > \frac{d-1}{64\varepsilon}$$

*training examples in the worst case.*

# Proving the Lower Bound

Let $X = \{x_1, \ldots, x_d\}$ be shattered by $\mathcal{C}$.

- Construct a pathological distribution that forces any algorithm to take many examples.
- $supp(\mathcal{D}) = X \Rightarrow$ w.l.o.g. $\mathcal{C} = \mathcal{C}(X)$, so $\mathcal{C}$ is a finite class, $|\mathcal{C}| = 2^d$.
- Choosing a $c$ from $\mathcal{C}$ is equivalent to tossing a fair coin $d$ times to determine the labeling on $X$.
- Suppose there is a learning algorithm $\mathcal{A}$ that uses at most $m = \left\lceil \frac{d-1}{64\varepsilon} \right\rceil$ training examples producing a hypothesis $h$.
- Want to show:
  $(\exists \mathcal{D} \text{ on } X)(\exists c \in \mathcal{C}) \left[\Pr_{S\sim\mathcal{D}^m}\left(\mathrm{Risk}_{\mathcal{D}}\left(\mathrm{h}, \mathrm{c}\right) > \varepsilon\right) > 1/15\right]$.

# Proving the Lower Bound

- Define $\mathcal{D}$ independently of $\mathcal{A}$:
$$\begin{cases} \Pr(x_1) & = & 1 - 16\varepsilon \\ \Pr(x_2) & = & \Pr(x_3) = \ldots = \Pr(x_d) = \frac{16\varepsilon}{d-1} \end{cases}$$
- Let $X' = \{x_2, x_3, \ldots, x_d\}$.
- Let $\mathrm{Risk}'_{\mathcal{D}}(h, c) = \Pr_{x \sim \mathcal{D}}(h(x) \neq c(x) \wedge x \in X')$.

Note that

$$\begin{aligned} \mathrm{Risk}_{\mathcal{D}}(h, c) &= \Pr_{x \sim \mathcal{D}}(h(x) \neq c(x)) \\ &\geq \Pr_{x \sim \mathcal{D}}(h(x) \neq c(x) \wedge x \in X') \\ &= \mathrm{Risk}'_{\mathcal{D}}(h, c) . \end{aligned}$$

- It is easier to prove $\Pr_{S \sim \mathcal{D}^m}\left(\mathrm{Risk}'_{\mathcal{D}}(\mathrm{h}, \mathrm{c}) > \varepsilon\right) > 1/15$.
  - But then the result follows from the above observation.

# Proving the Lower Bound

**Probabilistic argument:** Pick a random $c \in \mathcal{C}$ and show that $c$ is hard to learn for $\mathcal{A}$ with positive probability. This implies that there is at least one $c \in \mathcal{C}$ that is hard to learn for $\mathcal{A}$.

**Idea:** Argue that the sample $S$ containing $m$ iid examples from $\mathcal{D}$, will miss more than half of the points from $X'$.

- $h$ will be 'guessing' the labels for these points $\Rightarrow$ inevitable to have large risk under $\mathcal{D}$.
- Expected # of instances from $X'$ appearing in $S$:
  $\mu = \left[\frac{16\varepsilon}{d-1} \cdot (d-1)\right] \cdot \left(\frac{d-1}{64\varepsilon}\right) = \frac{d-1}{4}$.
- Markov $\Rightarrow$ Pr $\left(\text{\# of instances from } X' \text{ in } S \geq \frac{d-1}{2}\right) \leq \frac{\frac{d-1}{4}}{\frac{d-1}{2}} = 1/2$.
- Define the **bad event**

$$B \equiv S \text{ contains less than } \tfrac{d-1}{2} \text{ instances from } X' \,.$$

By the above,

$$\Pr_{S\sim\mathcal{D}^m}(B) = 1 - \Pr_{S\sim\mathcal{D}}\left(\text{\# instances from } X' \text{ in S} \geq \frac{d-1}{2}\right) \geq \frac{1}{2}\,. \qquad (1)$$

# Proving the Lower Bound

- $h$ is independent of $X' \setminus S$
- we pick $c \in \mathcal{C}$ at random

So, $h$ will make a mistake on each instance $x \in X' \setminus S$ with probability $1/2$.

- Each instance $x \in X' \setminus S$ contributes to $\text{Risk}'_{\mathcal{D}}(h, c)$ an amount of $\frac{1}{2} \cdot \frac{16\varepsilon}{(d-1)}$.
- When the bad event $B$ occurs, we have $|X' \setminus S| > \frac{d-1}{2}$.

This implies

$$\mathrm{E}_{c,S}\left[\text{Risk}'_{\mathcal{D}}(h, c) \ \mid \ B\right] > 4\varepsilon\,. \tag{2}$$

- By (1) and (2) we get a lower bound on $\mathrm{E}_{c,S}\left[\text{Risk}'_{\mathcal{D}}(h, c)\right]$:

$$\mathrm{E}_{c,S}\left[\text{Risk}'_{\mathcal{D}}(h, c)\right] \geq \mathrm{E}_{c,S}\left[\text{Risk}'_{\mathcal{D}}(h, c) \ \mid \ B\right] \cdot \Pr_S(B) > (4\varepsilon) \cdot (1/2) = 2\varepsilon\,.$$

(We used $\mathrm{E}[Y] = \sum_i \mathrm{E}[Y \mid A_i] \cdot \Pr(A_i)$, where $A_i$: finite or countable partition of the sample space.)

# Proving the Lower Bound

$\mathrm{E}_{c,S}\left[\mathrm{Risk}'_{\mathcal{D}}(h,c)\right] > 2\varepsilon \Longrightarrow (\exists c^\star \in \mathcal{C})\ \left[\mathrm{E}_S\left[\mathrm{Risk}'_{\mathcal{D}}(h,c^\star)\right] > 2\varepsilon\right]$.

- Take that $c^\star$ as the target concept.
- Show that $\mathcal{A}$ will be prone to produce an $h$ with large risk.

$\mathrm{Risk}'_{\mathcal{D}}(h,c) = \mathrm{Pr}_{x\sim\mathcal{D}}(h(x) \neq c(x) \wedge x \in X') \leq \mathrm{Pr}_{x\sim\mathcal{D}}(x \in X') = 16\varepsilon$.
So,

$$\mathrm{E}_S\left[\mathrm{Risk}'_{\mathcal{D}}(h,c) \;\middle|\; \mathrm{Risk}'_{\mathcal{D}}(h,c) > \varepsilon\right] \leq 16\varepsilon\,.$$

Therefore,

$$\begin{aligned}
2\varepsilon &< \mathrm{E}_S\left[\mathrm{Risk}'_{\mathcal{D}}(h,c)\right] \\
&= \mathrm{Pr}_S\left(\mathrm{Risk}'_{\mathcal{D}}(h,c) > \varepsilon\right)\cdot \mathrm{E}_S\left[\mathrm{Risk}'_{\mathcal{D}}(h,c) \;\middle|\; \mathrm{Risk}'_{\mathcal{D}}(h,c) > \varepsilon\right] \\
&\quad + (1 - \mathrm{Pr}_S\left(\mathrm{Risk}'_{\mathcal{D}}(h,c) > \varepsilon\right))\cdot \mathrm{E}_S\left[\mathrm{Risk}'_{\mathcal{D}}(h,c) \;\middle|\; \mathrm{Risk}'_{\mathcal{D}}(h,c) \leq \varepsilon\right] \\
&\leq \mathrm{Pr}_S\left(\mathrm{Risk}'_{\mathcal{D}}(h,c) > \varepsilon\right)\cdot(16\varepsilon) + (1 - \mathrm{Pr}_S\left(\mathrm{Risk}'_{\mathcal{D}}(h,c) > \varepsilon\right))\cdot(\varepsilon) \\
&= 15\varepsilon\cdot \mathrm{Pr}_S\left(\mathrm{Risk}'_{\mathcal{D}}(h,c) > \varepsilon\right) + \varepsilon\,.
\end{aligned}$$

In other words, $\mathrm{Pr}_S\left(\mathrm{Risk}'_{\mathcal{D}}(h,c) > \varepsilon\right) > \frac{1}{15}$.